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First-Passage and Extreme-Value Statistics of a Particle Subject to a Constant Force Plus a Random Force

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Abstract We consider a particle which moves on the x axis and is subject to a constant force, such as gravity, plus a random force in the form of Gaussian white noise. We analyze the statistics of first arrival at point x_1 of a particle which starts at x_0 with velocity v_0. The probability that the particle has not yet arrived at x_1 after a time t, the mean time of first arrival, and the velocity distribution at first arrival are all considered. We also study the statistics of the first return of the particle to its starting point. Finally, we point out that the extreme-value statistics of the particle and the first-passage statistics are closely related, and we derive the distribution of the maximum displacement $m = \max_t[x(t)]$.

Keywords Random acceleration · Random force · First passage · Extreme statistics · Stochastic process · Non-equilibrium statistics

1 Introduction

In this paper we consider a particle which moves on the x axis and is subject to both a constant force, such as gravity, and a random force in the form of Gaussian white noise. The Newtonian equation of motion is given by

$$\frac{d^2x}{dt^2} = g + \eta(t), \tag{1}$$

$$\langle\eta(t)\rangle = 0, \quad \langle\eta(t)\eta(t')\rangle = 2\Lambda\delta(t-t'), \tag{2}$$

where g is a constant.

Simple stochastic processes such as this are of both mathematical and physical interest. The approximately random collision forces experienced by a particular particle in a many particle system are often modelled by Gaussian white noise. Langevin's equation [1, 2] for the motion of a Brownian particle in a constant force field corresponds to (1) with an

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additional viscous damping term of the form $-\lambda dx/dt$ on the right-hand side. On setting $g=0$ in (1) and regarding t as a Cartesian coordinate instead of time, one may interpret the path $x(t)$ of the particle as a configuration of a semi-flexible polymer [3]. For several applications of the process (1) related to semi-flexible polymers and driven granular matter, see [3–6] and references therein.

In this paper we study first-passage properties [7] of the process (1). More precisely, we analyze the statistics of the first arrival at point x_1 of a particle which starts at x_0 with velocity v_0. Due to translational invariance no generality is lost in choosing x_1 to be the origin, and since we consider both positive and negative g, no generality is lost in choosing x_0 to be positive. Thus, "first passage" corresponds to the first exit of the particle from the positive x axis. If the initial velocity v_0 is positive, the particle must return to its initial position at least once before exiting from the positive x axis. Thus, in the limit $x_0 \searrow 0$ with $v_0 > 0$, the first-passage statistics reduces to the statistics of first return of the particle to its initial position. Clearly, first-passage statistics, as defined here, is the same as the statistics of absorption of a particle moving on the half line $x=0$ with an absorbing boundary at $x=0$.

First-passage properties of the random acceleration process, corresponding to (1) but without the constant term g on the right hand side, are derived or reviewed in Refs. [3, 5, 8–10]. In the remainder of this section we show how these results can be generalized to include the constant force. In Sect. 2 some statistical quantities of interest in connection with first passage are defined, and in Sect. 3 our explicit results are presented. In Sect. 4 we show that the extreme-value statistics [5, 11–14] and first-passage statistics of the process (1) are closely related. This is then used in deriving the distribution of the maximum displacement $m=\max_t[x(t)]$ of a particle which begins at the origin with velocity v_0.

With no loss of generality we replace the parameters g and Λ, introduced in (1) and (2), by $g \to \gamma = \pm 1$ and $\Lambda = 1$ throughout this paper, since this can be achieved by rescaling[1] the variables x and t. For $\gamma = -1$ and $\gamma = 1$, the constant force drives a particle on the positive x axis toward and away from the origin, respectively.

Integrating the equation of motion (1) yields

$$x(t) = x_0 + v_0 t + \frac{1}{2}\gamma t^2 + \int_0^t (t-t')\eta(t')dt', \tag{3}$$

which, together with properties (2) of the random force, implies the moments

$$\langle x\rangle = x_0 + v_0 t + \frac{1}{2}\gamma t^2, \qquad \langle (x-\langle x\rangle)^2\rangle = \frac{1}{3}t^3. \tag{4}$$

Thus, the contribution of the random force on the right side of (3) has typical size $t^{3/2}$. For large t the constant force is more important than the random force, but for small t the opposite is true.

It is convenient to define $P_\gamma(x,v;x_0,v_0;t)dxdv$ as the probability that the position and velocity of a particle, moving according to (1) with $g\to\gamma=\pm1$ and $\Lambda=1$, evolve from x_0, v_0 to values between x and $x+dx$, v and $v+dv$ in a time t without ever reaching $x=0$. The probability distribution P_γ satisfies the time-dependent Fokker-Planck equation [1, 2]

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial v} - \frac{\partial^2}{\partial v^2}\right) P_\gamma(x,v;x_0,v_0;t) = 0, \tag{5}$$

[1] In terms of the dimensionless variables $\xi = |g|^3\Lambda^{-2}x$, $\tau = |g|^2\Lambda^{-1}t$, (1) and (2) take the form $d^2\xi/d\tau^2 = \pm 1 + \zeta(\tau)$, $\langle\zeta(\tau)\rangle = 0$, $\langle\zeta(\tau)\zeta(\tau')\rangle = 2\delta(\tau-\tau')$.

with the initial condition

$$P_\gamma(x, v; x_0, v_0; 0) = \delta(x - x_0)\delta(v - v_0) \tag{6}$$

and the boundary condition

$$P_\gamma(0, v; x_0, v_0; t) = 0, \quad v > 0. \tag{7}$$

This boundary condition ensures that only trajectories which exit the positive x axis for the first time at time t contribute to $P_\gamma(0, v; x_0, v_0; t)$. Trajectories which leave the positive x axis at an earlier time and return to the positive x axis are excluded. Equation (7) is also the appropriate boundary condition for motion on the half line $x > 0$ with an absorbing boundary at $x = 0$.

In the absence of the constant force, the corresponding probability distribution $P_0(x, v; x_0, v_0; t)$ satisfies the same Fokker-Planck equation (5), initial condition (6), and boundary condition (7), except that the term $\gamma\partial P/\partial v$ in (5) is absent. This implies the relation

$$P_\gamma(x, v; x_0, v_0; t) = \exp\left[\frac{1}{2}\gamma(v - v_0) - \frac{1}{4}t\right] P_0(x, v; x_0, v_0; t) \tag{8}$$

between the distributions with and without the constant force, which is central to our work.

In a classic paper on the first-passage properties of a randomly accelerated particle, McKean [8] derived the exact propagator $P_0(0, -v; 0, v_0; t)$ for $v > 0$ and $v_0 > 0$, corresponding to a particle which leaves the origin with velocity v_0 and returns for the first time at time t with velocity $-v$ and speed v. His result and (8) imply

$$\begin{aligned} &P_\gamma(0, -v; 0, v_0; t) \\ &\quad = \frac{\sqrt{3}}{2\pi t^2} \exp\left[-\frac{1}{2}\gamma(v + v_0) - \frac{1}{4}t - (v^2 - vv_0 + v_0^2)/t\right] \operatorname{erf}\left(\sqrt{\frac{3vv_0}{t}}\right), \end{aligned} \tag{9}$$

where $\operatorname{erf}(z)$ denotes the standard error function [15, 16].

The Laplace transform

$$\tilde{P}_\gamma(x, v; x_0, v_0; s) = \int_0^\infty dt e^{-st} P_\gamma(x, v; x_0, v_0; t) \tag{10}$$

plays a central role in our work. Substituting (9) on the right-hand side, using the integral representation [15, 16] $\operatorname{erf}(z) = 2\pi^{-1/2} z \int_0^1 dy \exp(-z^2 y^2)$, and integrating over t with the help of Ref. [16], we obtain

$$\begin{aligned} &\tilde{P}_\gamma(0, -v; 0, v_0; s) \\ &\quad = \frac{3}{2\pi}(vv_0)^{1/2} e^{-\gamma(v+v_0)/2} \int_0^1 dy \exp\left[-(4s+1)^{1/2}(v^2 - vv_0 + v_0^2 + 3vv_0y^2)^{1/2}\right] \\ &\quad \times \left[(v^2 - vv_0 + v_0^2 + 3vv_0y^2)^{-3/2} + (4s+1)^{1/2}(v^2 - vv_0 + v_0^2 + 3vv_0y^2)^{-1}\right]. \end{aligned} \tag{11}$$

We will also need McKean's result [8] for the Laplace transform,

$$\begin{aligned} &\tilde{P}_\gamma(0, -v; 0, v_0; s) \\ &\quad = \frac{e^{-\gamma(v+v_0)/2}}{\pi^2 vv_0} \int_0^\infty d\mu\mu \frac{\sinh(\pi\mu)}{\cosh(\frac{1}{3}\pi\mu)} K_{i\mu}\big(\sqrt{(4s+1)}v\big) K_{i\mu}\big(\sqrt{(4s+1)}v_0\big), \end{aligned} \tag{12}$$

where $K_\mu(z)$ is a modified Bessel function [15, 16]. Expressions (11) and (12) are particularly convenient for numerical and analytical calculations, respectively, and both expressions are used below.

The exact solution $\tilde{P}_0(x, v; x_0, v_0; s)$ of the Fokker-Planck equation for random acceleration on the half line $x > 0$ with boundary condition (7), is given in Ref. [3], where it is derived from more general results of Marshall and Watson [9]. All of our results for first passage from an arbitrary initial point x_0 are based on this solution. Substituting it in (8) and setting $x = 0$, we obtain

$$\tilde{P}_\gamma(0, -v; x_0, v_0; s) = e^{-\gamma(v+v_0)/2} \int_0^\infty dF e^{-Fx_0} \phi_{s+1/4,F}(-v)\psi_{s+1/4,F}(v_0), \tag{13}$$

where

$$\psi_{s,F}(v) = F^{-1/6}\mathrm{Ai}\left(F^{1/3}v + F^{-2/3}s\right), \tag{14}$$

$$\phi_{s,F}(v) = \psi_{s,F}(v) - \frac{1}{2\pi}\int_0^\infty dG \frac{\exp[-\frac{2}{3}s^{3/2}(F^{-1}+G^{-1})]}{F+G}\psi_{s,G}(-v), \tag{15}$$

and $\mathrm{Ai}(z)$ is the Airy function [15]. Some important properties of the two set of basis functions $\psi_{s,F}(v)$ and $\phi_{s,F}(v)$ are discussed in Ref. [3]. For example, $\phi_{s,F}(v)$ vanishes identically for $v > 0$, so that (13) satisfies the boundary condition (7).

2 Statistical Quantities of Interest

The "survival probability" or probability that a particle with initial position and velocity x_0 and v_0 has not yet left the positive x axis after a time t is given by

$$Q_\gamma(x_0, v_0; t) = \int_{-\infty}^\infty dv \int_0^\infty dx\, P_\gamma(x, v; x_0, v_0; t). \tag{16}$$

According to (5), (7), and (16)

$$\frac{\partial}{\partial t} Q_\gamma(x_0, v_0; t) = -\int_0^\infty dv v P_\gamma(0, -v; x_0, v_0; t). \tag{17}$$

Thus, we interpret

$$v P_\gamma(0, -v; x_0, v_0; t) dv dt, \tag{18}$$

for $x_0 > 0$, as the probability that the particle reaches the origin for the first time at a time between t and $t + dt$ with speed between v and $v + dv$.

Several useful relations follow from this interpretation of the quantity (18). The survival probability defined by (16), its limiting value for $t \to \infty$, and its Laplace transform can be written in the form

$$Q_\gamma(x_0, v_0; t) = 1 - \int_0^t dt' \int_0^\infty dv v P_\gamma(0, -v; x_0, v_0; t'), \tag{19}$$

$$Q_\gamma(x_0, v_0; \infty) = 1 - \int_0^\infty dv v \tilde{P}_\gamma(0, -v; x_0, v_0; 0), \tag{20}$$

$$\tilde{Q}_\gamma(x_0, v_0; s) = \frac{1}{s}\left[1 - \int_0^\infty dv v \tilde{P}_\gamma(0, -v; x_0, v_0; s)\right]. \tag{21}$$

The mean time to exit the positive x axis for the first time is given by

$$T_\gamma(x_0, v_0) = \frac{\int_0^\infty dt\, t \int_0^\infty dv\, v P_\gamma(0, -v; x_0, v_0; t)}{\int_0^\infty dt \int_0^\infty dv\, v P_\gamma(0, -v; x_0, v_0; t)} \tag{22}$$

$$= -\left[1 - Q_\gamma(x_0, v_0; \infty)\right]^{-1} \lim_{s\to 0} \frac{\partial}{\partial s} \int_0^\infty dv\, v \tilde{P}_\gamma(0, -v; x_0, v_0; s). \tag{23}$$

Since a particle which begins at the origin with a negative velocity immediately moves onto the negative x avis, Q_γ and T_γ satisfy the boundary conditions

$$Q_\gamma(0, v_0; t) = 0, \quad v_0 < 0,\ t > 0, \tag{24}$$

$$T_\gamma(0, v_0) = 0, \quad v_0 < 0. \tag{25}$$

Finally, the probability that the speed of the particle on exiting from the positive x axis for the first time is between v and $v + dv$ is given by $G_\gamma(v; x_0, v_0)dv$, where

$$G_\gamma(v; x_0, v_0) = \frac{v \int_0^\infty dt\, P_\gamma(0, -v; x_0, v_0; t)}{\int_0^\infty dv\, v \int_0^\infty dt\, P_\gamma(0, -v; x_0, v_0; t)} = \frac{v \tilde{P}_\gamma(0, -v; x_0, v_0; 0)}{1 - Q_\gamma(x_0, v_0; \infty)}, \tag{26}$$

and the normalization $\int_0^\infty dv G_\gamma(v; x_0, v_0) = 1$ has been imposed.

3 Results

3.1 Limit $Q_\gamma(x_0, v_0; \infty)$ of the Survival Probability

The survival probability $Q_\gamma(x_0, v_0; t)$ introduced in the preceding section can, in principle, be evaluated for arbitrary t from (9), (13), and (19). However, this involves integrating over t' and v and, for $x_0 > 0$, inverting a Laplace transform to go from s to t, most of which must be performed numerically. In this section we consider the limiting value $Q_\gamma(x_0, v_0; \infty)$ or probability that in an infinite time the particle never leaves the positive x axis, which can be obtained analytically.

In the absence of a constant force, $Q_0(x_0, v_0, t)$ decays as $t^{-1/4}$ in the long-time limit [3, 8, 10]. Thus, $Q_0(x_0, v_0, \infty) = 0$, which means that the particle exits from the positive x axis in an infinite time with probability 1. As shown in (3), the constant force adds an extra term $\frac{1}{2}\gamma t^2$ to the displacement $x(t)$ of the randomly accelerated particle. In the case $\gamma = -1$ of a constant force toward the origin, the particle leaves the positive x axis sooner than without the constant force, so

$$Q_{-1}(x_0, v_0; \infty) = 0. \tag{27}$$

In the case $\gamma = 1$ of a constant force pushing the particle away from the origin, the corresponding probability $Q_1(x_0, v_0, \infty)$ does not vanish and is expected to increase as x_0 and v_0 increase. Recalling (24), substituting (12) and (13) into (20), and proceeding as described in the Appendix, we obtain

$$Q_1(0, v_0; \infty) = \begin{cases} 0, & v_0 < 0, \\ \operatorname{erf}\left(\sqrt{\tfrac{3}{2} v_0}\right), & v_0 > 0, \end{cases} \tag{28}$$

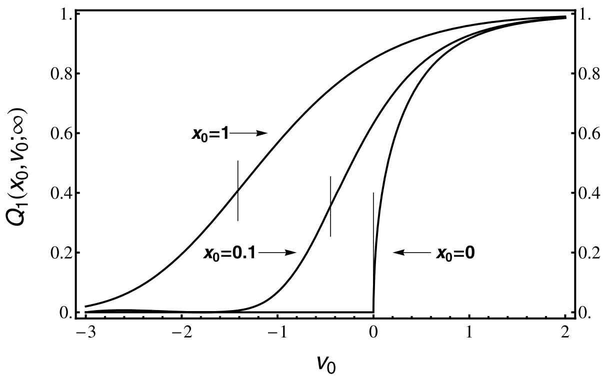

Fig. 1 Probability $Q_1(x_0, v_0; \infty)$, given by (28) and (29), that a particle subject to a random force plus a constant force pushing it away from the origin never leaves the positive x axis. If the random force is switched off, $Q_1(x_0, v_0; \infty)$ becomes a unit step function, as discussed in the second paragraph below (31). The short vertical lines in the figure indicate the values of v_0 at which the step function jumps from 0 to 1 for $x_0 = 0, 0.1, 1$

$$Q_1(x_0, v_0; \infty) = 1 - \frac{e^{-v_0/2}}{\sqrt{2\pi}} \times \int_0^\infty dF\, F^{-7/6} \exp\left(-\frac{1}{12F} - F x_0\right) \mathrm{Ai}\left(F^{1/3} v_0 + \frac{1}{4} F^{-2/3}\right). \quad (29)$$

As in (9) and (14), $\mathrm{erf}(z)$ denotes the error function, and $\mathrm{Ai}(z)$ is the Airy function, both defined as in Ref. [15]. We found it useful to make the change of variables $F = u^{-6}$ in evaluating the integrals in (29), (34), and (50) numerically with *Mathematica*.

Equations (28) and (29) imply the asymptotic behavior

$$Q_1(0, v_0; \infty) \approx \begin{cases} \left(\frac{6v_0}{\pi}\right)^{1/2}, & v_0 \searrow 0, \\ 1 - \left(\frac{2}{3\pi v_0}\right)^{1/2} e^{-3v_0/2}, & v_0 \to \infty, \end{cases} \quad (30)$$

$$Q_1(x_0, 0; \infty) \approx \begin{cases} \frac{2^{5/6} 3^{1/3}}{\Gamma\left(\frac{1}{3}\right)} x_0^{1/6}, & x_0 \searrow 0, \\ 1 - \left(\frac{3}{8\pi^2 x_0}\right)^{1/4} e^{-(2x_0/3)^{1/2}}, & x_0 \to \infty, \end{cases} \quad (31)$$

$$Q_1(x_0, v_0; \infty) \approx \begin{cases} \mathrm{erf}\left(\sqrt{\frac{3}{2}} v_0\right) + \left[\frac{\partial}{\partial x} Q_1(x, v_0; \infty)\right]_{x=0} x_0, & v_0 > 0,\ x_0 \searrow 0, \\ \frac{3^{3/2}}{\sqrt{2\pi}} \frac{x_0}{|v_0|^{5/2}} e^{-v_0/2 - |v_0|^3/9x_0}, & v_0 < 0,\ x_0 \searrow 0, \\ 1 - \left(\frac{3}{8\pi^2 x_0}\right)^{1/4} e^{-v_0 - (2x_0/3)^{1/2}}, & x_0 \to \infty. \end{cases} \quad (32)$$

In Fig. 1, $Q_1(x_0, v_0; t)$, as given by (28) and (29), is plotted as a function of v_0 for several values of x_0. Note that $Q_1(x_0, v_0; t)$ increases monotonically with increasing x_0 and v_0, as expected.

If the random force is switched off, the particle trajectory becomes $x(t) = x_0 + v_0 t + \frac{1}{2}\gamma t^2$, which never reaches the origin for $\gamma = 1$ and $v_0 > -\sqrt{2x_0}$. Thus, each of the smooth curves in Fig. 1 is replaced by the unit step function $Q_1(x_0, v_0; \infty) = \theta(v_0 + \sqrt{2x_0})$. The short vertical lines in Fig. 1 indicate the value of v_0 at which $Q_1(x_0, v_0; \infty)$ is discontinuous.

3.2 Mean First-Passage Time $T_\gamma(x_0, v_0)$

The $t^{-1/4}$ decay of the survival probability $Q_0(x_0, v_0, t)$ of a randomly accelerated particle [3, 8, 10] is so slow that the mean time of its first exit from the positive x axis, given by $T_0(x_0,v_0)=\int_0^\infty dt t[-\partial Q_0(x_0,v_0,t)/\partial t]$, is infinite. However, in the case $\gamma=-1$ of a constant force toward the origin in addition to the random force, the corresponding mean time $T_{-1}(x_0,v_0)$ is finite. Recalling (25), substituting (12), (13), (28), and (29) into (23), and proceeding as described in the Appendix, we obtain

$$T_{-1}(0,v_0)=\begin{cases}0, & v_0<0,\\ \sqrt{\frac{2v_0}{\pi}}e^{-v_0/2}+\left(\frac{2}{3}+v_0\right)\left[2-\operatorname{erfc}\left(\sqrt{\frac{1}{2}v_0}\right)\right] & \\ \qquad -\frac{2}{3}e^{3v_0/2}\operatorname{erfc}\left(\sqrt{2v_0}\right), & v_0>0,\end{cases} \tag{33}$$

$$\begin{aligned}T_{-1}(x_0,v_0)=&\ \frac{1}{8}\sqrt{\frac{3}{\pi}}\int_0^\infty dt t^{-3/2}\left(6x_0+2v_0t+t^2\right)\exp\left[-\frac{3}{4}\left(\frac{x_0+v_0t}{t^{3/2}}-\frac{1}{2}t^{1/2}\right)^2\right]\\ &-\frac{e^{v_0/2}}{2\pi}\int_0^\infty dF F^{-7/6}\exp\left(-\frac{1}{12F}-Fx_0\right)\mathrm{Ai}\left(F^{1/3}v_0+\frac{1}{4}F^{-2/3}\right)\\ &\times\left[\sqrt{6\pi}-\frac{\pi}{\sqrt{F}}\exp\left(\frac{1}{6F}\right)\operatorname{erfc}\left(\frac{1}{\sqrt{6F}}\right)\right],\end{aligned} \tag{34}$$

where $\operatorname{erfc}(z)=1-\operatorname{erf}(z)$ is the complementary error function [15].

Equations (33) and (34) imply the asymptotic behavior

$$T_{-1}(0,v_0)\approx\begin{cases}\left(\frac{18v_0}{\pi}\right)^{1/2}, & v_0\searrow 0,\\ 2v_0+\frac{4}{3}, & v_0\to\infty,\end{cases} \tag{35}$$

$$T_{-1}(x_0,0)\approx\begin{cases}\left(\frac{2}{\pi}\right)^{1/2}3^{5/6}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}x_0^{1/6}, & x_0\searrow 0,\\ (2x_0)^{1/2}+\frac{2}{3}, & x_0\to\infty.\end{cases} \tag{36}$$

The leading terms $2v_0$ and $(2x_0)^{1/2}$ in (35) and (36) for $x_0=0$, $v_0\to\infty$ and for $v_0=0$, $x_0\to\infty$ are easy to understand. According to the discussion following (4), the constant force is more important than the random force for large t, implying $x(t)\approx x_0+v_0t-\frac{1}{2}t^2$, which vanishes at $T_{-1}(x_0,v_0)\approx v_0+\sqrt{v_0^2+2x_0}$.

For small x_0 and v_0 the particle tends to reach the origin quickly, so $T_\gamma(x_0,v_0)$ is small. For short times the random force is more important than the constant force (see discussion following (4)) and primarily responsible for the asymptotic behavior $T_{\pm 1}(0,v_0)\sim v^{1/2}$ and $T_{-1}(x_0,0)\sim x^{1/6}$ in (35), (36), and (38). We note that these same power laws for small x_0 and v_0 appear in the mean first exit time [17, 18] of a randomly accelerated particle with initial position and velocity x_0, v_0 from the finite interval $0<x<L$.

In Fig. 2, $T_{-1}(x_0,v_0)$, as given by (33) and (34), is plotted as a function of v_0 for several values of x_0. As expected, $T_{-1}(x_0,v_0;t)$ increases monotonically as x_0 and v_0 increase.

As discussed just above (28), for $\gamma=1$ the probability $Q_1(x_0,v_0;\infty)$ that the particle never leaves the positive x axis is nonzero, in general, and it increases, as in (30) and (31), as v_0 and x_0 increase. However, the mean first exit time for those trajectories which do leave the positive x axis, defined by (23), is finite. From (12), (23), (25), and (28), we obtain

$$T_1(0,v_0)=\begin{cases}0, & v_0<0,\\ \frac{2}{3}-v_0+\left(\sqrt{\frac{6v_0}{\pi}}-\frac{2}{3}\right)\frac{e^{-3v_0/2}}{\operatorname{erfc}\left(\sqrt{\frac{3v_0}{2}}\right)}, & v_0>0,\end{cases} \tag{37}$$

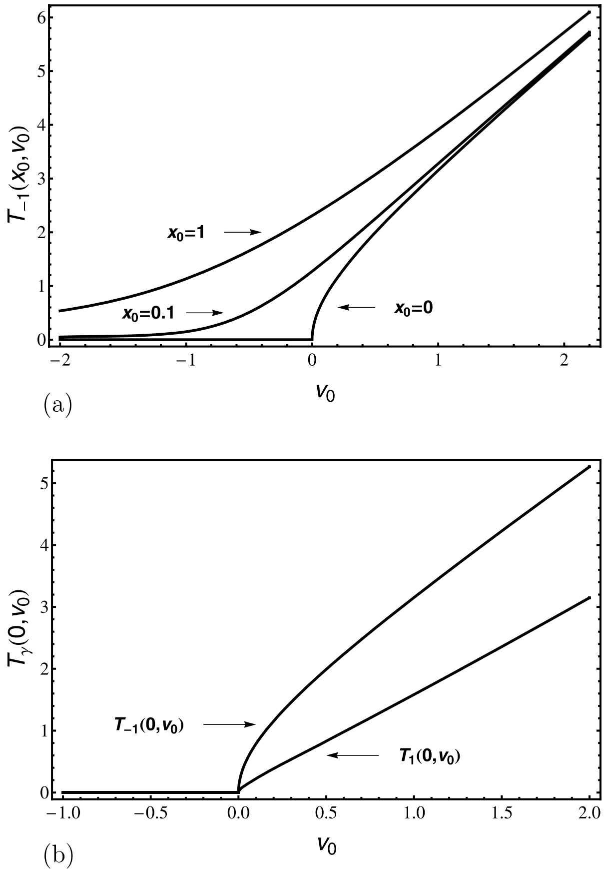

Fig. 2 (**a**) Mean time $T_{-1}(x_0, v_0)$ to exit the positive x axis for the first time for a particle subject to a random force plus a constant force toward the origin, given by (33) and (34). (**b**) Mean first exit times $T_{-1}(0, v_0)$ and $T_1(0, v_0)$, given in (33) and (37) for constant forces toward and away from the origin, respectively

which has the asymptotic behavior

$$T_1(0, v_0; \infty) \approx \begin{cases} \left(\frac{2v_0}{3\pi}\right)^{1/2}, & v_0 \searrow 0, \\ 2v_0 - \left(\frac{2\pi v_0}{3}\right)^{1/2} + \frac{5}{3}, & v_0 \to \infty. \end{cases} \quad (38)$$

For arbitrary x_0 and v_0, $T_1(x_0, v_0)$ follows from substituting (13) and (29) into (23). This leads to a lengthy expression, containing multiple integrals, which we were unable to simplify and omit here.

The results (33) and (37) for $T_{-1}(0, v_0)$ and $T_1(0, v_0)$ are compared in Fig. 2.

3.3 Distribution $G_\gamma(v; x_0, v_0)$ of the Particle Speed at First Passage

For arbitrary initial position x_0 and initial velocity v_0, the distribution $G_\gamma(v; x_0, v_0)$ of the particle speed on exiting from the positive x axis for the first time is determined by (13), (26), and (29). Here we restrict our attention to the case of a particle which begins at the origin with $v_0 > 0$. The distribution $G_\gamma(v; 0, v_0)$ of its speed on returning to the origin for the first time and leaving the positive x axis can be readily evaluated by substituting (11) and (28) into (26) and integrating over the variable y numerically. The results, for several values of v_0, are shown in Fig. 3.

Each of the curves in Fig. 3 has a single peak. As v_0 increases, the peak shifts to larger values of v, as expected, and becomes broader. The peak position and width correspond roughly to the mean speed $\langle v\rangle_\gamma$ at first return and the root-mean-square deviation $\sigma_\gamma = \langle (v-\langle v\rangle_\gamma)^2\rangle_\gamma^{1/2}$, where

$$\langle v^n\rangle_\gamma = \int_0^\infty dv\, v^n G_\gamma(v;0,v_0). \tag{39}$$

Using (12), (26), and (28) and the approach of the Appendix, we have calculated the first two moments of the speed at first return analytically, obtaining

$$\langle v\rangle_{-1} = v_0 + \frac{4}{3} + e^{-v_0/2} \times \left[\left(\frac{2v_0}{\pi}\right)^{1/2} - \left(v_0+\frac{2}{3}\right)e^{v_0/2}\mathrm{erfc}\left(\sqrt{\frac{v_0}{2}}\right) - \frac{2}{3}e^{2v_0}\mathrm{erfc}\left(\sqrt{2v_0}\right)\right], \tag{40}$$

$$\langle v^2\rangle_{-1} = v_0^2 + 2v_0 + 2\langle v\rangle_{-1}, \tag{41}$$

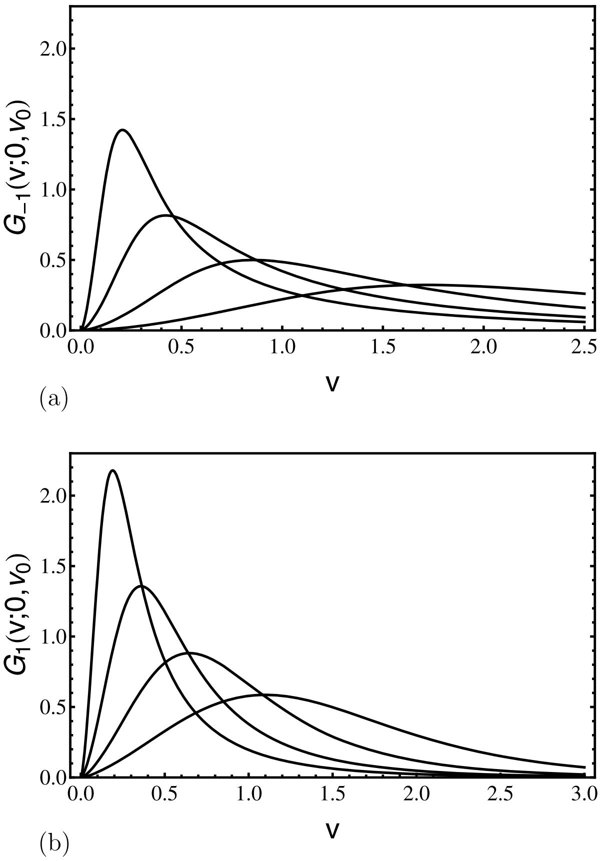

Fig. 3 Distribution $G_\gamma(v;0,v_0)$, given by (11), (26), and (28), of the particle speed v at first return, for $v_0 = 0.2$, 0.4, 0.8, and 1.6. The results in (**a**) and (**b**) are for constant forces directed toward and away from the origin, respectively. As v_0 increases, the peak becomes lower and broader and moves to the right

$$\langle v\rangle_1 = \frac{2}{3}\left(\frac{e^{-3v_0/2}}{\operatorname{erfc}\left(\sqrt{\frac{3v_0}{2}}\right)} - 1\right), \tag{42}$$

$$\langle v^2\rangle_1 = v_0^2 - 2v_0 + \frac{4}{3} - \left[\left(\frac{2v_0}{3\pi}\right)^{1/2}(v_0 - 5) + \frac{4}{3}\right]\left(\frac{3}{2}\langle v\rangle_1 + 1\right). \tag{43}$$

For large v_0 the average speed at first return and the root-mean-square deviation from the average have the asymptotic forms

$$\langle v\rangle_{-1} \approx v_0 + \frac{4}{3}, \tag{44}$$

$$\sigma_{-1} \approx \left(\frac{4}{3}v_0\right)^{1/2}, \tag{45}$$

and

$$\langle v\rangle_1 \approx \left(\frac{2\pi v_0}{3}\right)^{1/2}, \tag{46}$$

$$\sigma_1 \approx \left[\frac{1}{3}(8 - 2\pi)v_0\right]^{1/2}, \tag{47}$$

which are qualitatively consistent with the evolution of the curves in Fig. 3 as v_0 increases.

For large v_0 the constant force is more important than the random force, and $x(t) \approx x_0 + v_0 t + \frac{1}{2}\gamma t^2$. Thus, for $\gamma = -1$, the particle returns to its starting point with approximately the same speed it had initially. This is consistent with the asymptotic behavior in (44).

4 Extreme-Value Statistics

Consider a particle which begins at $x_0 = 0$ with velocity v_0 and moves according to (1) with $g \to \gamma = \pm 1$. At some time in the interval $0 < t < \infty$ the particle attains a maximum displacement $m = \max_t[x(t)]$. For large t, $x(t) \approx \frac{1}{2}\gamma t^2$, as follows from the discussion below (4). Thus, in the case $\gamma = 1$ of a constant force in the positive direction, $m = \infty$. In this section we consider the less trivial question of the maximum displacement m for $\gamma = -1$, and we derive the corresponding distribution $\mathcal{P}_{-1}(m, v_0)$. Distributions such as this play a central role in the field of extreme-value statistics [5, 11, 12]. The extreme-value statistics of a generalized Gaussian process that includes random acceleration as a special case is studied in Refs. [13, 14].

To derive the distribution $\mathcal{P}_{-1}(m, v_0)$, we begin by writing

$$\mathcal{P}_{-1}(m, v_0) = \frac{\partial}{\partial m}\mathcal{F}_{-1}1(m, v_0), \tag{48}$$

where $\mathcal{F}_{-1}(m, v_0)$ is the probability that, for a constant force in the *negative* direction, the displacement $x(t)$ of a particle which begins at the origin with velocity v_0 never exceeds m in the time interval $0 < t < \infty$. For $m < 0$, $\mathcal{F}_{-1}(m, v_0) = 0$, since the initial displacement $x_0 = 0$ already exceeds m. For $m > 0$, $\mathcal{F}_{-1}(m, v_0)$ is the same as the probability that, for a constant force in the *positive* direction, a particle with initial position m and initial velocity $-v_0$ never reaches the origin. This follows from the invariance of the probability under the coordinate transformation $x \to m - x$. Since this latter probability is precisely the survival probability $Q_1(m, -v_0; \infty)$ considered in Sects. 2 and 3,

$$\mathcal{F}_{-1}(m, v_0) = \theta(m) Q_1(m, -v_0; \infty), \tag{49}$$

where $\theta(m)$ is the standard step function.

Making use of (48) and (49) and the expressions for $Q_1(0, v_0; \infty)$ and $Q_1(x_0, v_0; \infty)$ in (28) and (29), we obtain

$$\begin{aligned} \mathcal{P}_{-1}(m, v_0) = {} & \theta(-v_0)\mathrm{erf}\left(\sqrt{\frac{3}{2}|v_0|}\right)\delta(m) \\ & + \theta(m)\frac{e^{v_0/2}}{\sqrt{2\pi}}\int_0^\infty dF\, F^{-1/6}\exp\left(-\frac{1}{12F} - Fm\right) \\ & \times \mathrm{Ai}\left(-F^{1/3}v_0 + \frac{1}{4}F^{-2/3}\right) \end{aligned} \tag{50}$$

for the extreme-value distribution. The distribution vanishes for $m < 0$ and is normalized so that $\int_{-\infty}^{\infty} dm \mathcal{P}_{-1}(m, v_0) = 1$, as follows from (48) and (49) and the boundary condition $Q_1(\infty, -v_0; \infty) = 1$. The first term on the right side of (50) has its origin in the non-zero probability $\mathrm{erf}\left(\sqrt{\frac{3}{2}|v_0|}\right)$ (see Sect. 3.1) that a particle which begins at the origin with $v_0 < 0$ never returns to the origin, in which case the maximum displacement m equals the initial value $x_0 = 0$.

The extreme-value distribution $\mathcal{P}_{-1}(m, v_0)$ is plotted as a function of m for several positive and negative values of v_0 in Figs. 4a and 4b, respectively. In the absence of the random force, $x = v_0 t - \frac{1}{2}t^2$, which implies $\mathcal{P}_{-1}(m, v_0) = \theta(-v_0)\delta(m) + \theta(v_0)\delta(m - \frac{1}{2}v_0^2)$. The random force broadens the delta functions, as seen in the figure.

For positive v_0, the peak in Fig. 4a shifts to larger values of m and becomes broader as v_0 increases, as expected. The mean value and the root-mean-square deviation vary as $\langle m\rangle \approx \frac{1}{2}v_0^2 + v_0$ and $\sigma \approx \left(\frac{2}{3}v_0^3\right)^{1/2}$ for large positive v_0. For large v_0 the constant force is more important than the random force, and the leading term in $\langle m\rangle$ equals the maximum displacement $\frac{1}{2}v_0^2$ of a particle subject only to the constant force.

In the results for $v_0 < 0$ in Fig. 4b, the vertical line at $m = 0$ represents the term proportional to $\delta(m)$ in (50). The most probable value of m, which maximizes $\mathcal{P}_{-1}(m, v_0)$, is zero for all negative v_0. The mean value of m is positive and, for v_0 negative and large in magnitude, $\langle m\rangle \approx \frac{4}{3}(2|v_0|/3\pi)^{1/2}e^{-3|v_0|/2}$, and $\langle m^2\rangle \approx \frac{32}{9}(2|v_0|^3/3\pi)^{1/2}e^{-3|v_0|/2}$.

5 Concluding Remarks

This completes our study of the first-passage and extreme-value statistics of the process (1). In closing we note that in a mathematical tour de force, Marshall and Watson [9] derived the Laplace transform $\tilde{P}_{g,\lambda}(x, v; x_0, v_0; s)$ of the solution to the Klein-Kramers equation with the absorbing boundary condition (7). The Klein-Kramers equation is the Fokker-Planck equation for the process [1, 2]

$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} = g + \eta(t), \tag{51}$$

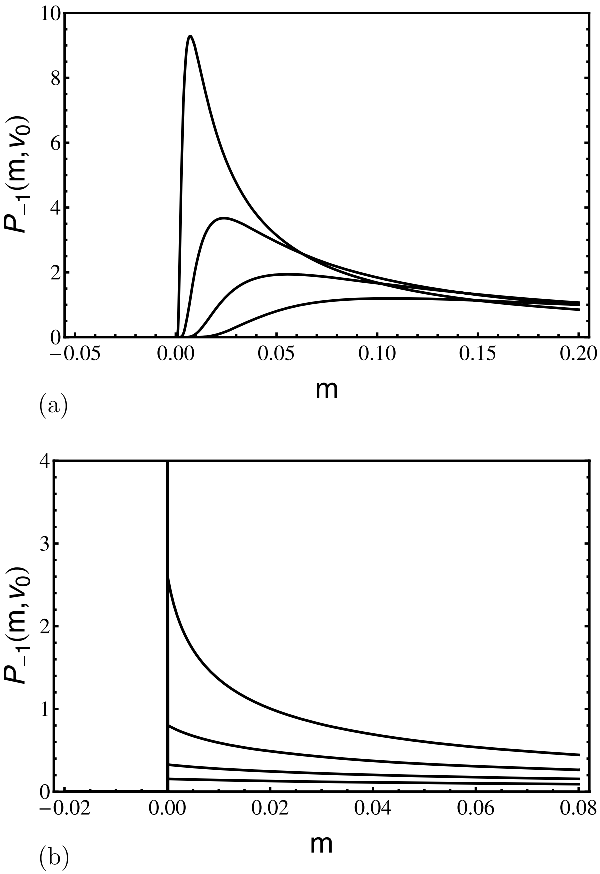

Fig. 4 Distribution $\mathcal{P}_{-1}(m, v_0)$, given by (50), of the maximum displacement m attained by a particle which begins at the origin with velocity v_0 and moves according to (1) with $g = -1$. The results in (**a**) are for $v_0 = 0.4$, 0.6, 0.8, and 1.0. As v_0 increases, the peak becomes lower and broader and moves to the right. The curves in (**b**) correspond, from top to bottom, to $v_0 = -0.4, -0.6, -0.8$, and -1. The vertical line at $m = 0$ represents the term in $P_{-1}(m, v_0)$ proportional to $\delta(m)$

which, unlike (1), includes viscous damping and plays a central role in the theory of Brownian motion. In principle, all of the first-passage and extreme-value properties we have considered follow, for this more general process, from the expression of Marshall and Watson for $\tilde{P}_{g,\lambda}(x, v; x_0, v_0; s)$, which, however, involves an infinite double sum over special functions and is difficult to work with.

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Appendix: Calculational Details

The quantities $Q_\gamma(0, v_0; \infty)$ and $T_\gamma(0, v_0)$, defined in (20) and (23), can both be expressed in terms of the integral

$$\int_0^\infty dv\, v\, \tilde{P}_\gamma(0, -v; 0, v_0; s)$$
$$= \frac{e^{-\gamma v_0/2}}{\pi^2 v_0} \int_0^\infty d\mu\, \mu \frac{\sinh(\pi\mu)}{\cosh(\frac{1}{3}\pi\mu)} F_\gamma(\mu, s) K_{i\mu}(\sqrt{(4s+1)}v_0), \qquad (52)$$

$$F_\gamma(\mu,s)=\int_0^\infty dv e^{-\gamma v/2}K_{i\mu}(\sqrt{(4s+1)}v), \tag{53}$$

where we have used the expression for $\tilde{P}_\gamma(0,-v;0,v_0;s)$ in (12). Evaluating $F_\gamma(\mu,s)$ with the help of the integral representation [15, 16]

$$K_{i\mu}(v)=\int_0^\infty dt\cos(\mu t)e^{-v\cosh t} \tag{54}$$

and substituting the result in (52), we obtain

$$\begin{aligned}&\int_0^\infty dvv\tilde{P}_\gamma(0,-v;x_0,v_0;s)\\&\quad=\frac{e^{-\gamma v_0/2}}{\pi v_0}\left(4s+1-\frac{1}{4}\gamma^2\right)^{-1/2}\\&\qquad\times\int_0^\infty d\mu\mu\frac{\sinh[\mu\arccos(\frac{1}{2}\gamma(4s+1)^{-1/2})]}{\cosh(\frac{1}{3}\pi\mu)}K_{i\mu}(\sqrt{(4s+1)}v_0).\end{aligned} \tag{55}$$

First we consider the quantity $Q_\gamma(0,v_0;\infty)$, defined in (20). For $\gamma=-1$ and $s=0$, the arccos in (55) equals $\frac{2}{3}\pi$. From (20) and (55) and the relation [16]

$$\int_0^\infty d\mu\mu\sinh(b\mu)K_{i\mu}(v_0)=\frac{1}{2}\pi v_0\sin b e^{-v_0\cos b}, \tag{56}$$

we obtain $Q_{-1}(0,v_0;\infty)=0$ for $v_0>0$, in agreement with (27).

For $\gamma=1$ and $s=0$, the arccos in (55) equals $\frac{1}{3}\pi$, and expression (28) for $Q_1(0,v_0;\infty)$, with $v_0>0$, follows from (20) and (55) and the relation

$$\int_0^\infty d\mu\mu\tanh\left(\frac{1}{3}\pi\mu\right)K_{i\mu}(v_0)=\frac{\sqrt{3}}{2}\pi v_0 e^{v_0/2}\text{erfc}\left(\sqrt{\frac{3v_0}{2}}\right), \tag{57}$$

which we derived with the help of the integral representation (54).

We now turn to $Q_\gamma(x_0,v_0;\infty)$ for $x_0\neq 0$. The result $Q_{-1}(x_0,v_0;\infty)=0$ was established in the paragraph containing (27). Expression (29) for $Q_1(x_0,v_0;\infty)$ follows from (13)–(15) and (20). The lengthy derivation will not be given here, but it is easy to see, with the help of Ref. [3], that the result (29) satisfies the appropriate Fokker-Planck equation $(v_0\partial_{x_0}+\gamma\partial_{v_0}+\partial_{v_0}^2)Q_\gamma(x_0,v_0;\infty)=0$ and to check, by numerical integration, that (28) and (29) agree for $x_0=0$.

The results (33) and (37) for $T_\gamma(0,v_0)$ may be derived by substituting (28) and (55) into (23) and using (56), (57) and some analogous integrals over μ which can be evaluated with the help of the integral representation (54). Expression (34) for $T_{-1}(x_0,v_0)$ follows from substituting (13)–(15) and (27) into (23), but the derivation is long and will not be given here. Making use of Ref. [3], we have confirmed that the result (34) satisfies the appropriate Fokker-Planck equation $(v_0\partial_{x_0}+\gamma\partial_{v_0}+\partial_{v_0}^2)T_\gamma(x_0,v_0;\infty)=-1$ and checked by numerical integration that (33) and (34) agree for $x_0=0$.

The moments $\langle v^n\rangle_\gamma$, defined by (26) and (39), can be expressed in terms in terms of the integral

$$\int_0^\infty dvv^{n+1}\tilde{P}_\gamma(0,-v;x_0,v_0;s)$$

$$= \frac{e^{-\gamma v_0/2}}{\pi v_0}\left(-2\frac{\partial}{\partial\gamma}\right)^n\left\{\left(4s+1-\frac{1}{4}\gamma^2\right)^{-1/2}\right.$$
$$\left.\times\int_0^\infty d\mu\mu\frac{\sinh[\mu\arccos(\frac{1}{2}\gamma(4s+1)^{-1/2})]}{\cosh(\frac{1}{3}\pi\mu)}K_{i\mu}\big(\sqrt{(4s+1)}v_0\big)\right\}. \quad (58)$$

This relation is the same as (55) except for the extra factor v^n introduced by applying $(-2\partial/\partial\gamma)^n$ to the quantity $F_\gamma(\mu, s)$ in (53). The steps leading from (58) to the results for $\langle v\rangle_\gamma$ and $\langle v^2\rangle_\gamma$ in (40)–(43) are very similar to the steps, described above, from (55) to the final expressions for $Q_\gamma(0, v_0; \infty)$ and $T_\gamma(0, v_0)$.

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